# The Laurent Expansion for a Nearly Singular Matrix

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ABSTRACT

Let $A_0$, $A_1$ be $n \times n$ matrices of complex numbers and let $E^n$ be the vector space of $n \times 1$ matrices of complex numbers. Let $N_1 = \{x \in E^n | A_1 x = 0\}$, $N_{0,-1} = \{0\} \subset E^n$, and for $k \geqslant -1$ define $R_{1k} = A_1 N_{0k}$ and $N_{0k+1} = \{x \in E^n | A_0 x \in R_{1k}\}$. In any case $\mu = \min\{k \geqslant -1 | N_{0,k+1} = N_{0k}\}$ exists and $\mu \geqslant 0$ or $\mu = -1$ according as $A_0$ is singular or not. The main result presented is the following: There exists $\delta > 0$ such that the matrix $A_0 + zA_1$ is invertible for all complex numbers $z$ such that $0 < |z| < \delta$ if and only if $N_1 \cap N_{0k} = \{0\}$ for all $k \geqslant 0$. Moreover, if this condition holds, then there exist $n \times n$ matrices $Q_k$ such that $(A_0 + zA_1)^{-1} = \sum_{k=-\mu-1}^{\infty} z^k Q_k$, the series converging for $0 < |z| < \delta$ for some $\delta > 0$, and $Q_{-\mu-1} \neq 0$.

## 1. INTRODUCTION

Let $A_0$ and $A_1$ be $n \times n$ matrices of complex numbers. We are interested here in the invertibility of the matrix $A_0 + zA_1$ for complex numbers $z \neq 0$. This requires that $\det(A_0 + zA_1)$, as a polynomial in $z$, be not identically zero and thus when $(A_0 + zA_1)^{-1}$ exists its elements are rational functions of $z$. Hence, for some $\delta > 0$, if $0 < |z| < \delta$, then

$$(A_0 + zA_1)^{-1} = z^{-m}Q_{-m} + \cdots + z^{-1}Q_{-1} + Q_0 + zQ_1 + \cdots, \qquad (1.1)$$

where the $n \times n$ matrices $Q_k$ are independent of $z$ and $Q_{-m} \neq 0$ for some $m \geqslant 0$. If $A_0$ is nonsingular, we may write $(A_0 + zA_1)^{-1} = A_0^{-1}(I + zA_1A_0^{-1})^{-1}$ and $m = 0$ since $(I + zK)^{-1} = I - zK + z^2K^2 - \cdots$ in some

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neighborhood of $z = 0$. The interesting case then is when $A_0$ is singular so that $m > 0$, and it is for this case that $A_0 + zA_1$ is "nearly singular" when $z$ is near zero.

If $A_0$ is nonsingular, we may also write $(A_0 + zA_1)^{-1} = z^{-1}A_0^{-1}(z^{-1}I + A_1A_0^{-1})^{-1}$, so in this case the expansion of $(A_0 + zA_1)^{-1}$ in a deleted neighborhood of $z = 0$ is related to the expansion of the resolvent operator $(\zeta I - K)^{-1}$ of $K = - A_1A_0^{-1}$ about $\zeta = \infty$. Thus many of the lemmas and definitions which appear below are analogous to corresponding ones in the spectral analysis theory of a linear operator $K$ on a linear space. (See, for example, [1, Chapter 5].) Since we are primarily concerned here with the case when $A_0$ is singular, we have had to establish appropriate modifications of such standard results.

In this paper we present what might be called an intrinsic condition which is necessary and sufficient that $(A_0 + zA_1)^{-1}$ have a valid Laurent expansion of the form (1.1). The parameter $m$ and the matrices $Q_k$ are characterized in terms of certain subspaces of $E^n$ exhibiting the interaction between $A_0$ and $A_1$. Here $E^n$ denotes the vector space of $n \times 1$ matrices $x$ over the complex numbers, and we interpret a $n \times n$ matrix $B$ as describing or effecting a linear transformation $x \to Bx$ of $E^n$ into $E^n$. In the problem at hand the ranges $R_i = \{A_ix | x \in E^n\}$ and null spaces $N_i = \{x \in E^n | A_ix = 0\}$, $i = 0, 1$, are important of course. They do not tell the whole story, however, and we must introduce two sequences of related subspaces which play the fundamental role.

## 2. THE BASIC SUBSPACES AND A NECESSARY CONDITION

In the following definition the fundamental sequences of related subspaces are presented.

DEFINITION 2.1. Let $N_{0,-1} = \{0\}$, the trivial subspace of $E^n$, and for $k \geqslant -1$ define

$$R_{1k} = A_1N_{0k}, \qquad N_{0,k+1} = \{x \in E^n | A_0x \in R_{1k}\}. \tag{2.1}$$

Let $R_{0,-1} = E^n$ and for $k \geqslant -1$ define

$$N_{1k} = \{x \in E^n | A_1x \in R_{0k}\}, \qquad R_{0,k+1} = A_0N_{1k}. \tag{2.2}$$

Of course we mean, for example, $A_0N_{1k} = \{A_0x | x \in N_{1k}\}$, and it is evident inductively that all the sets $R_{ik}$, $N_{ik}$ defined by (2.1) and (2.2) are subspaces of $E^n$. We note that $R_{1,-1} = \{0\}$ so $N_{00} = N_0$, and $N_{1,-1} =$

$E^n$ so $R_{00} = R_0$. The statements in the next lemma are easy consequences of Definition 2.1, and we prove only (2.4) to illustrate.

LEMMA 2.1. *For all integers* $k \geqslant 0$,

$$N_0 \subset N_{0k} \subset N_{0,k+1}; \qquad R_{1k} \subset R_{1,k+1} \subset R_1; \tag{2.3}$$

$$N_1 \subset N_{1,k+1} \subset N_{1k}; \qquad R_{0,k+1} \subset R_{0k} \subset R_0; \tag{2.4}$$

*if* $x \in N_{0,k}$, *then*

$$A_0 x = A_1 \tilde{x} \quad \textit{for some} \quad \tilde{x} \in N_{0,k-1}. \tag{2.5}$$

*Proof of* (2.4). Since $A_1 N_1 = \{0\} \subset R_{0k}$ for any $k \geqslant 0$, then $N_1 \subset N_{1k}$ for $k \geqslant 0$. Now obviously $N_{10} \subset N_{1,-1} = E^n$, so $R_{01} = A_0 N_{10} \subset A_0 N_{1,-1} = R_{00} = R_0$ whence also $N_{11} \subset N_{10}$. Thus $R_{0,k+1} \subset R_{0k}$ holds for $k = 0$. Assuming it for some $k \geqslant 0$, we then get $N_{1,k+1} \subset N_{1k}$ from the first of (2.2), and thus $R_{0,k+2} = A_0 N_{1,k+1} \subset A_0 N_{1k} = R_{0,k+1}$. Thus the monotonicity of $N_{1k}$ and $R_{0k}$ is true by induction.

Assertion (2.5) enables us to obtain a necessary condition for invertibility of $A_0 + zA_1$ as formulated in the following theorem. An obvious analog of (2.5) is also valid, but we have no need for it in our proofs.

THEOREM 2.1. *If* $A_0 + zA_1$ *is invertible for some* $z \neq 0$, *then* $N_1 \cap N_{0k} = \{0\}$ *for all* $k \geqslant 0$. ($N_1 \cap N_{0,-1} = \{0\}$ *trivially*.)

*Proof.* Let $z \neq 0$ be such that $A_0 + zA_1$ is invertible and suppose $k \geqslant 0$ is such that $N_1 \cap N_{0k} \neq \{0\}$. Moreover, we assume $k$ is the least such index so that $N_1 \cap N_{0,k-1} = \{0\}$. Now let $x \in N_1 \cap N_{0k}$, $x \neq 0$. We note first that $k > 0$ for, if $k = 0$, then $A_0 x = A_1 x = 0$ and $(A_0 + zA_1)x = 0$ with $x \neq 0$, thus contradicting the invertibility of $A_0 + zA_1$. We denote $x_k = x$ and then by (2.5) there exist $x_j \in N_{0j}, j = 0, 1, \ldots, k-1$, such that $A_0 x_{j+1} = A_1 x_j$. If we define $\xi = \sum_{j=0}^{k} (-z)^j x_j$, then

$$\begin{aligned}(A_0 + zA_1)\xi &= \sum_{j=0}^{k} (-z)^j A_0 x_j - \sum_{j=0}^{k} (-z)^{j+1} A_1 x_j \\ &= A_0 x_0 - (-z)^{k+1} A_1 x_k = 0\end{aligned}$$

since $x_0 \in N_0$ and $x_k = x \in N_1$. But $A_0 + zA_1$ is invertible, so $\xi = 0$ and $x = x_k = -(-z)^{-k}(x_0 - zx_1 + \cdots + (-z)^{k-1} x_{k-1}) \in N_{0,k-1}$ by (2.3).

Thus $x \in N_1 \cap N_{0,k-1}$, so $N_1 \cap N_{0,k-1} \neq \{0\}$, which contradicts the choice of $k$; therefore the theorem is proved.

For convenience in later references we shall denote the condition of interest by D; thus

$$\text{(D)} \qquad N_1 \cap N_{0k} = \{0\}, \qquad k = 0, 1, 2, \ldots .$$

In the next section we establish some further properties of the spaces $N_{ik}$ and $R_{ik}$ and some consequences of condition D. In Section 4 we prove, in effect, the converse of Theorem 2.1.

## 3. PROPERTIES OF THE BASIC SUBSPACES

Because of the monotonicity properties, (2.3) and (2.4), there are in fact only a finite number of distinct subspaces $N_{ik}$ and $R_{ik}$. Their dimensions are integer-valued, bounded monotone functions of $k$ so they must be constant from some value of $k$ on, and the sequences of subspaces then also are constant from some value of $k$ on. In fact, each is strictly monotone up to the beginning of the constant tail of the sequence.

LEMMA 3.1. *If* $N_{0,k+1} = N_{0k}$, *then* $N_{0,k+j} = N_{0k}$ *and* $R_{1,k+j} = R_{1k}$ *for* $j = 1, 2, \ldots$ . *If* $R_{0,k+1} = R_{0k}$, *then* $R_{0,k+j} = R_{0k}$ *and* $N_{1,k+j} = N_{1k}$ *for* $j = 1, 2, \ldots$ .

*Proof.* Suppose $N_{0,k+1} = N_{0k}$ so that $N_{0,k+j} = N_{0k}$ for some $j \geqslant 1$. Then also $R_{1,k+j} = A_1 N_{0,k+j} = A_1 N_{0k} = R_{1k}$ by (2.1) and, moreover, $N_{0,k+j+1} = N_{0,k+1} = N_{0k}$, whence $R_{1,k+j+1} = R_{1k}$. Thus the first statement in the lemma is true by induction, and the second can be shown similarly.

The remarks immediately preceding this lemma justify the following definitions

$$\mu = \min\{k \geqslant -1 | N_{0,k+1} = N_{0k}\}, \tag{3.1}$$

$$\lambda = \min\{k \geqslant -1 | R_{0,k+1} = R_{0k}\}, \tag{3.2}$$

since the sets involved here are not empty. Lemma 3.1 implies that these sets consist of all $k \geqslant \mu$ and all $k \geqslant \lambda$, respectively. We note here that the nonsingularity of $A_0$ is equivalent to $\mu = -1$ and also to $\lambda = -1$.

Thus we assume in what follows that $\mu \geqslant 0$ and $\lambda \geqslant 0$, although the results are usually valid also in case $\mu = \lambda = -1$. It is convenient, though, to be able to characterize $\mu$ by $N_{0,\mu+1} = N_{0,\mu} \neq N_{0,\mu-1}$ and to characterize $\lambda$ by $R_{0,\lambda+1} = R_{0\lambda} \neq R_{0,\lambda-1}$.

LEMMA 3.2. *For all integers $k \geqslant 0$, $j \geqslant 0$ the following hold:*

$$A_0(N_{0k} \cap N_{1,j-1}) = R_{1,k-1} \cap R_{0j}, \tag{3.3}$$

$$A_1(N_{0k} \cap N_{1,j-1}) = R_{1k} \cap R_{0,j-1}, \tag{3.4}$$

$$A_1 N_{1k} = R_1 \cap R_{0k}. \tag{3.5}$$

*Hence also the following hold:*

$$\dim N_{0k} \cap N_{1,j-1} = \dim N_0 \cap N_{1,j-1} + \dim R_{1,k-1} \cap R_{0j}, \tag{3.6}$$

$$\dim N_{0k} \cap N_{1,j-1} = \dim N_1 \cap N_{0k} + \dim R_{1k} \cap R_{0,j-1}, \tag{3.7}$$

$$\dim N_{1k} = \dim N_1 + \dim R_1 \cap R_{0k}. \tag{3.8}$$

*Proof.* Because the proofs of these are similar, we merely illustrate. Thus, for example, if $x \in N_{0k} \cap N_{1,j-1}$, it is clear from Definition 2.1 that $A_0 x \in R_{1,k-1} \cap R_{0j}$. On the other hand, if $y \in R_{1,k-1} \cap R_{0j}$, then $y = A_0 x$ for some $x \in N_{1,j-1}$ and, since also $A_0 x = y \in R_{1,k-1}$, then $x \in N_{0k}$ as well. Thus (3.3) holds and (3.6) follows from this since $N_0 \subset N_{0k}$, so the null space of $A_0$ on $N_{0k} \cap N_{1,j-1}$ is $N_0 \cap N_{1,j-1}$.

Besides the general results (3.6) and (3.7) we will need especially the case $j = 0$ in which $N_{1,j-1} = E^n$. Thus

$$\dim N_{0k} = \dim N_0 + \dim R_{1,k-1} \cap R_0, \qquad k \geqslant 0, \tag{3.9}$$

$$\dim N_{0k} = \dim N_1 \cap N_{0k} + \dim R_{1k}, \qquad k \geqslant 0. \tag{3.10}$$

One additional relation of an analogous nature will be needed. It arises from $R_{0,k+1} = A_0 N_{1k}$ in Definition 2.1. Thus

$$\dim N_{1k} = \dim N_0 \cap N_{1k} + \dim R_{0,k+1}. \tag{3.11}$$

LEMMA 3.3. *For all integers $k \geqslant 0$,*

$$\dim(R_{1k} + R_0) = \dim R_{1k} - \dim N_{0,k+1} + n, \tag{3.12}$$

$$\dim(R_{0k} + R_1) = \dim R_{0k} - \dim N_{1k} + n. \tag{3.13}$$

*Proof.* These relations follow easily from (3.9) and (3.8), respectively.

LEMMA 3.4. *If condition* D *holds, then*

$$\dim R_{1k} = \dim N_{0k}, \qquad k = 0, 1, \ldots, \tag{3.14}$$

$$R_{1k} + R_0 = E^n, \qquad \text{if and only if} \qquad k \geqslant \mu. \tag{3.15}$$

*Proof.* Under condition D ($N_1 \cap N_{0k} = \{0\}$ for $k \geqslant 0$) we see that (3.10) reduces to (3.14), so in (3.12) we have $\dim(R_{1k} + R_0) = \dim N_{0k} - \dim N_{0,k+1} + n$. By (3.1) and (2.3) we see that this is less than $n$ if $k \leqslant \mu - 1$ and equal to $n$ if $k \geqslant \mu$ so (3.15) must hold.

LEMMA 3.5. *Let $k \geqslant 0$, let $S_0$ be a subspace of $E^n$ such that $E^n = R_{1k} \oplus S_0$, and define $M_0 = \{x | A_1 x \in S_0\}$. If condition* D *holds, then $E^n = N_{0k} \oplus M_0$.*

*Proof.* If $x \in N_{0k} \cap M_0$, then $A_1 x \in R_{1k} \cap S_0$. Then $A_1 x = 0$, so $x \in N_1 \cap N_{0k}$ and $x = 0$ if D holds. Hence $N_{0k} + M_0 = N_{0k} \oplus M_0$. Note that $N_1 \subset M_0$ and, since $A_1 M_0 = R_1 \cap S_0$, then $\dim R_1 \cap S_0 = \dim M_0 - \dim N_1$. Now $R_{1k} \subset R_1$, so $R_1 = R_1 \cap (R_{1k} \oplus S_0) = R_{1k} \oplus (R_1 \cap S_0)$. Hence, by (3.14),

$$\begin{aligned} n - \dim N_1 = \dim R_1 &= \dim R_{1k} + \dim R_1 \cap S_0 \\ &= \dim N_{0k} + \dim M_0 - \dim N_1. \end{aligned}$$

Thus $\dim(N_{0k} \oplus M_0) = n$ and the lemma is proved.

LEMMA 3.6. *Let $k \geqslant 0$, let $M_0$ be a subspace of $E^n$ such that $E^n = N_{0k} \oplus M_0$, and define $S_1 = A_0 M_0$. If* D *holds, then*

$$\dim(R_{1k} + S_1) = n - \dim M_0 \cap N_{0,k+1} \tag{3.16}$$

*and*

$$E^n = R_{1k} \oplus S_1 \quad \text{if and only if} \quad k \geqslant \mu. \tag{3.17}$$

*Proof.* Since $N_0 \subset N_{0k}$ for $k \geqslant 0$, then $N_0 \cap M_0 = \{0\}$ so $\dim S_1 = \dim M_0$. We easily see that $A_0(M_0 \cap N_{0,k+1}) = S_1 \cap R_{1k}$, so we have, similarly, $\dim S_1 \cap R_{1k} = \dim M_0 \cap N_{0,k+1}$. Thus, by (3.14), $\dim(R_{1k} +$

$S_1) = \dim N_{0k} + \dim M_0 - \dim M_0 \cap N_{0,k+1}$, and (3.16) follows from $E^n = N_{0k} \oplus M_0$. Now since $N_{0k} \subset N_{0,k+1}$ we see that $\dim M_0 \cap N_{0,k+1} = 0$ if and only if $N_{0,k+1} = N_{0k}$, which is equivalent to $k \geqslant \mu$. Thus $E^n = R_{1k} + S_1$ if and only if $k \geqslant \mu$ and, when this holds, $S_1 \cap R_{1k} = A_0\{0\} = \{0\}$ so the sum is direct and (3.17) is proved.

LEMMA 3.7. *If* D *holds, then for all integers* $k \geqslant 0$,

$$R_{1\mu} + R_{0k} = E^n, \tag{3.18}$$

$$N_{0\mu} + N_{1k} = E^n. \tag{3.19}$$

*Proof.* From (3.15) we see that (3.18) holds for $k = 0$, so we let $k$ be any nonnegative integer for which it holds. Then there is a subspace $S_0 \subset R_{0k}$ such that $E^n = R_{1\mu} \oplus S_0$ and we define $M_0 = \{x | A_1 x \in S_0\}$. Clearly $M_0 \subset N_{1k}$, so $S_1 = A_0 M_0 \subset R_{0,k+1}$. By Lemma 3.5 we have $E^n = N_{0\mu} \oplus M_0$, whence by Lemma 3.6 we have $E^n = R_{1\mu} \oplus S_1 \subset R_{1\mu} + R_{0,k+1}$. Thus (3.18) also holds with $k$ replaced by $k + 1$, so it holds for all integers $k \geqslant 0$. We may get (3.19) from parts of the proof of (3.18). Thus $E^n = N_{0\mu} \oplus M_0 \subset N_{0\mu} + N_{1k}$.

LEMMA 3.8. *If* D *holds, then* $N_0 \cap N_{1k} = \{0\}$ *if and only if* $k \geqslant \lambda$.

*Proof.* From (3.18) it follows that $R_1 + R_{0k} = E^n$ for all $k \geqslant 0$. Thus by (3.13) we have

$$\dim R_{0k} = \dim N_{1k}, \qquad k = 0, 1, \ldots \tag{3.20}$$

if D holds. Using this in (3.11), we conclude that

$$\dim R_{0,k+1} = \dim R_{0k} - \dim N_0 \cap N_{1k}. \tag{3.21}$$

By (2.4), Lemma 3.1, and the definition of $\lambda$ in (3.2), it is clear from (3.21) that $N_0 \cap N_{1k} = \{0\}$ if and only if $k \geqslant \lambda$.

LEMMA 3.9. *If* D *holds, then, for all integers* $k \geqslant 0$, $j \geqslant 0$,

$$\dim N_{0,k+1} \cap N_{1,j-1} = \sum_{t=j-1}^{j+k} \dim N_0 \cap N_{1t}. \tag{3.22}$$

*Proof.* By (3.6) and (3.7) we have

$$\dim N_{0,k+1} \cap N_{1,j-1} = \dim N_0 \cap N_{1,j-1} + \dim N_{0k} \cap N_{1j}$$

when D holds. Using this to substitute in the last term, we get

$$\dim N_{0,k+1} \cap N_{1,j-1}$$
$$= \dim N_0 \cap N_{1,j-1} + \dim N_0 \cap N_{1j} + \dim N_{0,k-1} \cap N_{1,j+1}.$$

Repeating this substitution, we finally arrive at (3.22).

LEMMA 3.10. *If* D *holds, then* $\mu = \lambda$.

*Proof.* We first specialize (3.22) to the case $j = 0$. Thus, since $N_{1,-1} = E^n$,

$$\dim N_{0,k+1} = \dim N_0 + \sum_{t=0}^{k} \dim N_0 \cap N_{1t},$$

from which it is clear that

$$\dim N_{0,k+1} = \dim N_{0k} + \dim N_0 \cap N_{1k}.$$

(Note that this is also valid for $k = -1$.) Since $N_{0k} \subset N_{0,k+1}$, we see that $N_{0,k+1} = N_{0k}$ if and only if $N_0 \cap N_{1k} = \{0\}$. By Lemma 3.8 we then see that $N_{0,k+1} = N_{0k}$ if and only if $k \geqslant \lambda$. But by (3.1) this is equivalent to $\mu = \lambda$.

LEMMA 3.11. *If* D *holds, then* $E^n = N_{0\mu} \oplus N_{1\mu}$ *and* $E^n = R_{1\mu} \oplus R_{0\mu}$. *Moreover,* $A_1$ *maps* $N_{0\mu}$ *one-to-one onto* $R_{1\mu}$, $A_0$ *maps* $N_{1\mu}$ *one-to-one onto* $R_{0\mu}$, *and the inverses of these maps are linear transformations.*

*Proof.* This is trivially true if $\mu = -1$, so we suppose $\mu \geqslant 0$. Then in (3.22) we take $k = \mu - 1$ (note that (3.22) holds trivially for $k = -1$) and $j = \mu + 1$ and we get $\dim N_{0\mu} \cap N_{1\mu} = \sum_{t=\mu}^{2\mu} \dim N_0 \cap N_{1t} = 0$ by Lemmas 3.8 and 3.10. Thus with $k = \mu$ in (3.19) we have $E^n = N_{0\mu} \oplus N_{1\mu}$. From (3.7) we have $\dim R_{1\mu} \cap R_{0\mu} = \dim N_{0\mu} \cap N_{1\mu} = 0$ since $N_1 \cap N_{0\mu} = \{0\}$ by condition D. Thus with $k = \mu$ in (3.18) we have $E^n = R_{1\mu} \oplus R_{0\mu}$. Now $R_{1\mu} = A_1 N_{0\mu}$ by (2.1), and since $N_1 \cap N_{0\mu} = \{0\}$ this mapping from $N_{0\mu}$ to $R_{1\mu}$ is one-to-one. By (2.2) we have $A_0 N_{1\mu} = R_{0,\mu+1} = R_{0\mu}$ and by Lemmas 3.8 and 3.10 we have $N_0 \cap N_{1\mu} = \{0\}$, so this mapping from $N_{1\mu}$ to $R_{0\mu}$ is also one-to-one. The linearity of the inverses follows, of course, from that of the direct mappings.

## 4. SUFFICIENCY OF CONDITION D AND CONSTRUCTION OF THE LAURENT SERIES

Throughout this section we shall assume that condition D holds even when this is not mentioned explicitly. The inverses of the maps mentioned in Lemma 3.11 may be extended in various ways to the complementary spaces $R_{0\mu}$ and $R_{1\mu}$, respectively, so as to define linear transformations on $E^n$ into $E^n$. In particular, by virtue of Lemma 3.11, two $n \times n$ matrices $Q_0$ and $Q_{-1}$ are uniquely defined by the following rules.

DEFINITION 4.1.

$$Q_0 y = \begin{cases} 0, & \text{if} \quad y \in R_{1\mu} \\ x, & \text{if} \quad y \in R_{0\mu} \quad \text{and} \quad y = A_0 x \quad \text{with} \quad x \in N_{1\mu}, \end{cases} \tag{4.1}$$

$$Q_{-1} y = \begin{cases} 0, & \text{if} \quad y \in R_{0\mu} \\ x, & \text{if} \quad y \in R_{1\mu} \quad \text{and} \quad y = A_1 x \quad \text{with} \quad x \in N_{0\mu}. \end{cases} \tag{4.2}$$

We now develop some basic properties of the matrices $Q_0$ and $Q_{-1}$ in relation to $A_0$ and $A_1$.

LEMMA 4.1. *The matrix $Q_0 A_0$ effects the projection onto $N_{1\mu}$ along $N_{0\mu}$, and $Q_{-1} A_1$ the projection onto $N_{0\mu}$ along $N_{1\mu}$. Consequently,*

$$Q_0 A_0 + Q_{-1} A_1 = I. \tag{4.3}$$

*Proof.* If $x \in N_{0\mu}$, then $A_0 x \in R_{1\mu}$ since $N_{0\mu} = N_{0,\mu+1}$, so by (4.1) we see that $Q_0 A_0 x = 0$. If $x \in N_{1\mu}$, then $A_0 x \in R_{0,\mu+1} = R_{0\mu}$ so $Q_0 A_0 x = x$ by (4.1). Thus $Q_0 A_0$ projects onto $N_{1\mu}$ along $N_{0\mu}$ since $E^n = N_{0\mu} \oplus N_{1\mu}$. A similar argument shows that $Q_{-1} A_1$ projects onto $N_{0\mu}$ along $N_{1\mu}$ and (4.3) then follows.

LEMMA 4.2. *For any integers $k \geqslant 0$ and $j \geqslant 0$,*

$$Q_{-1} A_0 (N_{0k} \cap N_{1,j-1}) = N_{0,k-1} \cap N_{1j}. \tag{4.4}$$

*Proof.* By (3.3) we see that (4.4) is equivalent to

$$Q_{-1} (R_{1,k-1} \cap R_{0j}) = N_{0,k-1} \cap N_{1j}. \tag{4.5}$$

Now suppose $y \in R_{1,k-1} \cap R_{0j}$. Then $y = A_1 x$ for some $x \in N_{0,k-1} \subset N_{0\mu}$

and, since $R_{1,k-1} \subset R_{1\mu}$, then $x = Q_{-1}y$ by (4.2). Since also $A_1 x = y \in R_{0j}$, then $x \in N_{1j}$ so $Q_{-1}y \in N_{0,k-1} \cap N_{1j}$. On the other hand, if $x \in N_{0,k-1} \cap N_{1j}$, then $y = A_1 x \in R_{1,k-1} \cap R_{0j}$ and $x = Q_{-1}y$ by (4.2).

DEFINITION 4.2. For $k = 1, 2, 3, \ldots$ define

$$Q_{-(k+1)} = (-Q_{-1}A_0)^k Q_{-1}. \tag{4.6}$$

LEMMA 4.3. *For all integers $k \geqslant 1$,*

$$Q_{-k}A_0 + Q_{-(k+1)}A_1 = 0. \tag{4.7}$$

*Proof.* By (4.6) and (4.3) we have

$$Q_{-2}A_1 = -Q_{-1}A_0Q_{-1}A_1 = -Q_{-1}A_0 + Q_{-1}A_0Q_0A_0.$$

Lemma 4.1 and the definitions of $\mu$ and $Q_{-1}$ imply that $Q_{-1}A_0Q_0A_0 = 0$, and we see that (4.7) holds for $k = 1$. Completion of the induction proof is easy.

LEMMA 4.4. $Q_{-(\mu+1)} \neq 0$, *but* $Q_{-k} = 0$ *for* $k \geqslant \mu + 2$.

*Proof.* We first show that

$$Q_{-(k+1)}E^n = N_{0,\mu-k} \cap N_{1,k-1}, \qquad k = 0, 1, \ldots, \mu. \tag{4.8}$$

Indeed this is true for $k = 0$ by (4.2) since $N_{1,-1} = E^n$, $A_1$ maps $N_{0\mu}$ onto $R_{1\mu}$, and $E^n = R_{1\mu} \oplus R_{0\mu}$. Furthermore, if (4.8) holds for some $k$, such that $0 \leqslant k \leqslant \mu - 1$, then using (4.6) and (4.4) we find that it also holds with $k$ replaced by $k + 1$, so (4.8) is established. Now, if we set $k = \mu$ in (4.8), we get $Q_{-(\mu+1)}E^n = N_0 \cap N_{1,\mu-1}$. By Lemma 3.8 and the fact that $\mu = \lambda$ we see that $Q_{-(\mu+1)} \neq 0$. However, $A_0Q_{-(\mu+1)}E^n = \{0\}$, so $Q_{-(\mu+2)} = -Q_{-1}A_0Q_{-(\mu+1)} = 0$, and from (4.6) we see that $Q_{-k} = 0$ for all $k \geqslant \mu + 2$.

DEFINITION 4.3. For $k = 1, 2, 3, \ldots$ define

$$Q_k = (-Q_0A_1)^k Q_0. \tag{4.9}$$

LEMMA 4.5. *For all integers $k \geqslant 0$,*

$$Q_{k+1}A_0 + Q_kA_1 = 0. \tag{4.10}$$

*Proof.* By (4.9) and (4.3) we have

$$Q_1 A_0 = -Q_0 A_1 Q_0 A_0 = -Q_0 A_1 + Q_0 A_1 Q_{-1} A_1.$$

For any $x \in E^n$ we have $Q_{-1} A_1 x \in N_{0\mu}$ by Lemma 4.1. Hence $A_1 Q_{-1} A_1 x \in R_{1\mu}$, so $Q_0 A_1 Q_{-1} A_1 x = 0$ by (4.1). Thus $Q_0 A_1 Q_{-1} A_1 = 0$, so (4.10) holds for $k = 0$. Now, if it holds for some $k \geqslant 0$, then by (4.9)

$$Q_{k+2} A_0 = (-Q_0 A_1) Q_{k+1} A_0 = (-Q_0 A_1)(-Q_k A_1) = -Q_{k+1} A_1,$$

so (4.10) holds for all $k \geqslant 0$ by induction.

We can now prove the sufficiency of condition D for the invertibility of $A_0 + zA_1$ in some deleted neighborhood of $z = 0$.

THEOREM 4.1. *If $N_1 \cap N_{0k} = \{0\}$ for all integers $k \geqslant 0$, then*

$$Q(z) = \sum_{k=-(\mu+1)}^{\infty} z^k Q_k \tag{4.11}$$

*converges for $0 < |z| < \delta$ for some $\delta > 0$. Moreover, for $z$ in this range $Q(z) = (A_0 + zA_1)^{-1}$.*

*Proof.* From (4.9) we see that, for $k \geqslant 0$ and for any matrix norm (in the operator sense),

$$\|z^k Q_k\| = |z|^k \|Q_k\| \leqslant |z|^k \cdot \|Q_0 A_1\|^k \cdot \|Q_0\|. \tag{4.12}$$

Thus, if $\|Q_0 A_1\| \neq 0$, then the series (4.11) converges for $0 < |z| < \delta = \|Q_0 A_1\|^{-1}$ while, if $\|Q_0 A_1\| = 0$, the same is true with any $\delta > 0$. Now for $N > 0$ we have

$$\left(\sum_{k=-(\mu+1)}^{N} z^k Q_k\right)(A_0 + zA_1)$$

$$= z^{-(\mu+1)} Q_{-(\mu+1)} A_0 + \sum_{k=-\mu}^{N} z^k (Q_k A_0 + Q_{k-1} A_1) + z^{N+1} Q_N A_1$$

$$= I + z^{N+1} Q_N A_1$$

by (4.7), (4.3), (4.10), and Lemma 4.4. Now, if $0 < |z| < \delta$, we see from (4.12) that $z^{N+1} Q_N A_1 \to 0$ as $N \to \infty$, so for such $z$ we have $Q(z)(A_0 + zA_1) = I$ and $Q(z) = (A_0 + zA_1)^{-1}$ as claimed.

## 5. SOME EXAMPLES

Let $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $N_0 = N_{00}$ is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $N_1$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $N_1 \cap N_{00} = \{0\}$. But $R_{10} = A_1 N_{00}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $N_{01}$ is all of $E^2$ and $N_1 \cap N_{01} \neq \{0\}$. This illustrates that condition D may not hold even though $N_1 \cap N_0 = \{0\}$. Note that $N_{10} = E^2$ in this example, so $R_{01} = R_{00} \neq E^2$ and $\lambda = 0$. Since $N_{00} \neq N_{01}$, we have $\mu = 1$. (See Lemma 3.10.)

As another example, take $A_1 = I$ and

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case $(A_0 + zA_1)^{-1} = z^{-3}A_0{}^2 - z^{-2}A_0 + z^{-1}I$ for all $z \neq 0$. The basic subspaces can easily be determined by the reader. Note that here $\mu + 1 = \operatorname{rank} A_1$, but with $\tilde{A}_0 = A_0{}^2$ we have $(\tilde{A}_0 + zA_1)^{-1} = - z^{-2}\tilde{A}_0 + z^{-1}I$ and for this $\mu + 1 < \operatorname{rank} A_1$. Thus $\mu$ is not determined merely by the rank of $A_1$.

Finally consider the case

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have

$$(A_0 + zA_1)^{-1} = z^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for which $\mu + 1 = 1$. For this $A_0$ and the previous $\tilde{A}_0$ we have $\operatorname{rank} A_0 = \operatorname{rank} \tilde{A}_0$, and we see that $\mu$ is not determined merely by the rank of $A_0$ either.

## REFERENCE

1 A. E. Taylor, *Introduction to Functional Analysis*, John Wiley & Sons, Inc., New York, 1958.

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